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The Physics of the Tumbling Spring

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An interesting study of the dynamics of a continuous structure is that of wave motion along a helical spring. A device, consisting solely of such a spring, is currently for sale in toy shops, with its primary appeal based on the fact that it can be made to "walk down stair steps as though it were alive." When the device is demonstrated there is invariably curiosity shown as to the reasons for its actions. The following discussion attempts to examine its behavior and some of the factors involved.

A typical toy of this sort (Fig. 1) consists of some 78 turns of helical coil spring of outside diameter 7.3 cm and total length 6.3 cm, constructed by winding edge-wise rectangular steel wire of cross section 0.3×0.081 cm. The total mass of the spring is about 300 gm, so that its mass per unit length is about 48 gm/cm. If the force required to stretch a short section of the spring is measured, it is found that about 11,000 dynes will produce a deflection equal to the original length of the section. In other words, the spring constant per unit length is about 11,000 dynes. The spring is wound so that its restoring force is zero when adjacent turns are touching.

The action of the spring in the course of performing its specialty is best explained by reference to Fig. 2. These sketches represent the spring as it progresses down one stair step. The motion is started with the spring on the upper step [Fig. 2(a)], by grasping the A-end, quickly raising and rotating it about the diameter of one coil as an axis, and allowing it to fall to the right and onto the lower step [Fig. 2(b)]. That part of the spring on the upper step rapidly moves so as to pile itself upon the lower step [Fig. 2(c)]. The velocity with which the individual turns move depends upon the distance the spring has been stretched, and that, in turn, depends upon the height of the step. The action that takes place immediately following the rise of the B-end from the upper step is determined by its velocity. If it is moving relatively slowly (corresponding to a small step), it may strike the top of the pile of coils already on the lower step [Fig. 2(d)], after which it moves on to the right [Fig. 2(e)] so as to fall to the next lower step. If the velocity of the B-end is greater (corresponding to a larger step), the end will fly completely past the remainder of the spring without striking it and fall directly to the next lower step. In either case, the action is continuous and self-sustaining with the spring progressing from step to step. The action is very smooth and uniform if the height of the step is not too great—say not over 5 to 10 cm. If the height is too great, the action becomes violent and erratic. Moreover, instead of moving along a straight-line path, the spring is apt to veer from side to side at each step.

The same general type of motion can also be produced if an inclined plane is used, rather than discrete stair steps. Perhaps the motion in

Fig. 1. Typical spring made by edge-winding 78 turns of rectangular steel wire, 0.3×0.081 cm.

Fig. 2. Successive configurations of spring as it moves from a higher step to a lower step.
either case may be more accurately described as “tumbling” than as “walking.”

Experimentation with different heights of steps and different conditions of operation shows that the time required for the spring to traverse one step is very nearly constant and independent of these variations. It is found to be just under 0.5 sec for the spring described. Evidently this time is primarily a characteristic of the spring alone. Owing to this fact, it seems possible to analyze the motion as a compressional wave (or more accurately, an “extensional” wave) set up along the longitudinal axis of the spring. A disturbance produced by displacing one end of the spring travels to the other end, where it is reflected, with this process continuing at each step. The characteristics of the spring are such that the displacement produced by the wave may be large compared with the length of the spring itself. For a uniform spring with a linear longitudinal axis, the phase velocity of such a wave is known to be

\[ v = (k/D)^{1/2}, \]  

where \( k \) is the spring constant per unit length, and \( D \) is the mass per unit length of the spring.

With the values of these constants quoted previously for the spring, this velocity is found to be 15 cm/sec. If the dissipation in the spring is negligible, an arbitrary disturbance will progress from one end of the spring to the other with this velocity and will require a time of 0.42 sec. This figure is based on the assumption that the spring is straight, whereas actually it is almost semicircular during most of the action. However, the time predicted in this way, on the basis of a traveling wave, is only slightly less than the time of about 0.5 sec required for the spring to move down one step. A discussion, presented later, about the action when the disturbance reaches the end of the spring may help to explain the difference in these two times.

Before presenting a detailed analysis of these end effects, further examination of the general behavior of the spring may be helpful. When the spring is on the upper step, it possesses a certain potential energy with reference to the lower step. In addition, when the A-end is tossed over and onto the lower step, kinetic energy is given the spring. The extensional wave resulting from this initial displacement travels the length of the spring with the definite velocity previously mentioned. When the B-end finally flies over, it is traveling along a circular path to the right. If the velocity of the B-end is relatively small, it may strike the coils already on the lower step, with an attendant loss of energy. The B-end still has a component of velocity to the right, but if the motion is to continue, the energy now possessed by the B-end must be sufficient to cause it to move far enough so it may fall freely to the next lower step. If the velocity of the B-end is somewhat greater, it may fly entirely past the stationary coils without striking them. Such action requires that the centrifugal forces exceed the forces due to the stretching of the spring and to gravity, which tend to collapse the spring.

Energy is lost during normal operation by internal friction in the spring, in the impact of successive turns on one another, and in the impact of the spring upon the surface of each step. If operation is to be uniform and continuous, the energy added to the spring by its progression from the higher to the lower step must be just sufficient to replace that lost during this progression. With the spring described, a step of about 3 cm is just enough to maintain steady operation. The material of which the surface of the step is made is relatively unimportant in this connection, so long as it is rigid enough to prevent the spring from tipping.

While, at first glance, the almost semicircular shape of the longitudinal axis of the spring seems to complicate the action, actually it very cleverly allows gravitational forces to add necessary

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energy to the system. The spring cannot maintain its longitudinal axis horizontal if only the two ends are supported, because the lateral stiffness is very slight. The semicircular shape may be maintained with support only at the ends. Even if a linear longitudinal axis were possible, there would be no mechanism for adding the energy lost during each progression of the wave along the spring.

The semicircular shape does complicate considerably the mathematical analysis of the action of the spring, since the motion involves linear displacement and simultaneous rotation about at least two axes. However, considerable insight into the behavior may be obtained through the study of two analogous problems relating to a helical spring with a linear longitudinal axis. The results of these problems may be applied to the tumbling spring in order to determine its motion as the B-end flies from the higher step to the lower step.

For the first of these analogous problems, consider the case of helical spring resting on a horizontal plane surface with the longitudinal axis of the spring vertical. A force is applied to the upper end of the spring so as to stretch it along its axis [Fig. 3(a)]. Opposing this applied force are the restoring force of the spring and the gravitational forces acting on each particle of the spring. The gravitational force on an element of wire of length $dz$ is $\sigma g dz$, where $\sigma$ is the linear density of the wire composing the spring, and $g$ is the acceleration due to gravity. The force $P_1$ [Fig. 3(b)] transmitted through the lower end of the element is $\rho dx/ds$, while the force $P_1$ transmitted through the upper end is $\rho [(dx/ds) + (d^2x/ds^2)ds]$. The constant $\rho$ relates the force to the angle of deflection $dx/ds$. These expressions involve the assumptions that the angle of deflection is small and that initially the turns are very close together. For equilibrium, the difference of these latter forces must equal the gravitational force, or

$$ \rho d^2x/ds^2 = \sigma g. $$

The solution to this equation, subject to the condition that at $s = 0$, both $dx/ds = 0$ and $x = 0$, is

$$ x = (\sigma g/\rho) \cdot \frac{1}{2} s^2. $$

In Eq. (2), $x$ is measured parallel to the longitudinal axis of the spring, while $s$ is measured along the wire of the spring. Both $x$ and $s$ are measured from the point where the spring has no deflection.

When the tumbling spring is in motion, its longitudinal axis is almost semicircular. However, as an approximate treatment, the analysis in terms of the linear axis may be applied. At the instant when the spring is in such a position as that shown in Fig. 2(c), with each end supported, it may be considered as though it were made of two shorter springs with their upper ends pulled up and joined together. If the distance through which each spring is pulled is known, the length of wire in that portion of the spring may be found from Eq. (2).

For example, suppose that the spring has the configuration of Fig. 4(b), with its longitudinal axis forming a semicircle of diameter $a$. If it is considered as two shorter springs with their ends joined, each spring is stretched a mean distance of $\frac{1}{2} a$. Therefore, the length of wire per short spring involved in this deflection is, by Eq. (2), $S'(2\pi a/2ag)^3$. The total length of wire involved in the deflection of the entire spring is twice this, or $S = (2\pi a/2ag)^3$. The number of turns of the entire spring involved is $N_1 = S/(\pi b)$, where $b$ is the mean diameter of each turn of the spring [Fig. 4(a)].

It has already been observed that a disturbance travels along the spring with a velocity equal to $(k/D)^3$. Thus, the time for the disturbance to travel the entire length of the spring ($N_2$ turns) is $h(D/k)^3$, where $h$ is the length of the spring when it is not stretched [Fig. 4(a)]. The time required for the disturbance, involving $N_2$ turns, to pass any given turn is

$$ (N_2/N_3) h(D/k)^3 = S h(D/k)^3/(\pi b N_2). $$

Therefore, the mean velocity with which a given turn moves
from one side of the disturbance to the other is $\frac{1}{2} \pi a$ divided by this time, or $\pi abN_0(k/D)^4/2S_1h$. The maximum velocity with which the given turn moves is greater than this latter value, since the turn must be at rest just before and after the disturbance passes. In fact, because the deflection is parabolic in nature [Eq. (2)], the maximum velocity is just twice the average velocity. This is true because the slope of a chord from the origin to any point on a parabola of the type $y = x^2$ is just half the slope of the parabola at the point where it is cut by the chord. Thus, the maximum velocity with which a given turn of the spring moves as the disturbance passes it is $\pi abN_0(k/D)^4/S_1h$.

When the disturbance has traveled the length of the spring and reached the B-end, this end rises from the upper step with a velocity of the value just calculated. As the B-end rises it begins to traverse a path that is approximately a semicircle, whose center is at the top of the stationary part of the spring [Fig. 4(b)]. The radius of this semicircle is $a$. The angular velocity of the B-end as it rises is, therefore,

$$\omega_2 = \pi abN_0(k/D)^4/S_1h.$$  \hspace{1cm} (3)

For the second problem, analogous to that of the tumbling spring, consider a helical spring pivoted at one end and whirling with constant angular velocity in a plane perpendicular to the axis of the pivot (Fig. 5). The spring sweeps through a circular region having its center at the pivot and its circumference traversed by the free end of the spring. Each element of the spring is subject to a force tending to pull it away from the pivot, and a restoring force tending to pull it toward the pivot. If gravitational forces are neglected, the two forces must balance if the element is in equilibrium. The total centrifugal force at any point $z$ along the spring, due to elements of the spring farther from the pivot than $z$, is

$$F_z = \int_z^S \sigma \omega^2 x dx.$$

where $S$ is the total length of wire in the spring, and $\omega$ is the angular velocity of rotation. Both $x$ and $s$ are measured from the pivot, as in the first problem where they were used. The force transmitted through any element of the spring is proportional to the change in its deflection angle $dx/ds$, referred to the angle $(dx/ds)_0$, which exists with no forces acting. This force may be written as $F = \rho[(dx/ds) - (dx/ds)_0]$, where the angle of deflection is assumed small. For equilibrium, this spring force, evaluated at the point $z$, must equal the centrifugal force at $z$, or

$$\int_z^S \sigma \omega^2 x dx = \rho[(dx/ds)_0 - A],$$

where $A = (dx/ds)_0$. It may be verified readily that the following is a solution to this equation:

$$x = \frac{A \sin(cs)}{c \cos(cS)} + B[\cos(cs) + \tan(cS)\sin(cs)],$$

where $c^2 = \omega^2(\sigma/\rho)$, and $B$ is an arbitrary constant. If the condition that $x = 0$ at $s = 0$ is imposed, then $B = 0$, and

$$x = A \sin(cs) / c \cos(cS).$$  \hspace{1cm} (4)

Evidently, if $cS < \frac{1}{2} \pi$, the system is stable, but if $cS = \frac{1}{2} \pi$, the denominator equals zero, and $x$ becomes indefinitely large. Actually, of course, $x$ would increase only until Hooke's law is no longer applicable to the spring. This condition defines a critical angular velocity

$$\omega_c = \pi(\rho/\sigma)^{1/2}$$  \hspace{1cm} (5)

for which the centrifugal force exceeds the restoring force of the spring.

This whirling spring is approximated by the tumbling spring at the time when the B-end flies over, even though the axis of the spring is not linear. At this time, the disturbance on the spring has traveled to the B-end and only half of the total disturbance remains on the spring. The B-end is moving with its maximum velocity, and the value of $S$ for Eq. (5) is $S' = \frac{1}{2} S_1$. Also, the linear density $\sigma$ of the wire may be written as $\sigma = hD/N_0\pi b$. Similarly, the spring constant $k$, in terms of the angular deflection of the spring wire, may be written as $k = N_0\pi bh^2/h$. If these latter three relations are substituted into Eq. (5),
the result is:

$$\omega_e = \pi^2 b N_0 (k/D)^{1/2} S \beta.$$  \hspace{1cm} (6)

A comparison of Eqs. (3) and (6) shows that the angular velocity with which the B-end rises from the upper step is the same as the critical angular velocity which will cause the spring to stretch itself indefinitely. Therefore, under these conditions the B-end will rise from the upper step, fly over the stationary part of the spring without falling upon it, and then fall to the next lower step. This is the type of action that is observed when the spring is tumbling normally down a series of steps of adequate height.

Admittedly, a number of serious approximations are contained in these calculations. Perhaps the most difficult one to justify is the application of relations set up for a spring with a linear axis to the case of the spring with its axis semicircular. The gravitational force, which would tend to collapse the spring as the B-end flies over, has been neglected. The velocity with which the B-end first rises is not the maximum velocity, although this maximum velocity should be attained shortly. Equations set up for the spring at rest have been applied to its behavior when in motion. The mathematics assumes that Hooke’s law holds for the spring, which experimentally is found not to be true where the deflections are large. Despite these approximations, the calculations strikingly predict that the angular velocity of the B-end is just that required to prevent its falling inward.

Numerical values for some of the quantities appearing in the preceding calculations are of interest for the tumbling spring whose constants were given at the beginning of the discussion. When the spring is displaced in typical fashion, its axis forms a semicircle whose diameter \( a \) is about 10 cm. With this deflection, it is found that 47 turns \( (N_1) \) of the spring are involved, and that 0.25 sec is required for the disturbance to pass any given turn. The maximum velocity with which a given turn moves is 125 cm/sec. The maximum angular velocity \( \omega_e \), and also the critical angular velocity \( \omega_c \), is 12.5 rad/sec. The total time required for the spring to move from one step to the next must be the time required for the disturbance to move from one end of the spring to the other—that is, 0.42 sec, as found from wave-motion theory—plus some fraction of the time needed for the B-end to rise from the upper step and fly to the lower step. The experimental time of progression—about 0.5 sec—agrees very well with that predicted by the analysis.

One further problem in connection with the spring may be examined. This is the question of how the time required for the spring to progress from one step to the next will change if all of the dimensions of the spring are scaled up or down by the same factor. It is assumed that the spring is always made of the same material and in the same way. This question may be answered by examining Eq. (1), which gives the velocity \( v \) of the traveling wave on the spring as \( (k/D)^{1/2} \).

The total mass of the spring varies as \( k^3 \), where \( k \) is the length of the spring, so that the mass per unit length \( D \) varies as \( k^3 \). In order to determine the variation of the spring constant per unit length, \( k \), it is necessary to study the force required to stretch a helical spring. The force \( F \) which will produce a given extension \( e \) is known to be:

$$\frac{F}{e} = \frac{C J}{S (k/b)^2 \cos^2 \alpha},$$

where \( C \) is the shear modulus of the material composing the spring, \( J \) is the polar moment of inertia of the cross section of the wire in the spring, \( S \) is the total length of wire, \( b \) is the diameter of a coil, and \( \alpha \) is the winding angle. The number of turns in the spring is \( S/b \). The length \( h \) of the spring is \( 2b S \sin \alpha / b \). Thus, \( k \) is given by the equation

$$k = \frac{F e}{4 S \sin \alpha \cos^2 \alpha}.$$  \hspace{1cm} (7)

The quantity \( J \) varies as \( k^3 \), while \( b \) varies as \( k \), so that \( k \) varies as \( k^3 \).

Since both \( k \) and \( D \) vary as \( k^3 \), the velocity \( v \) is unchanged by a change in the scale of the spring. The distance that a disturbance must move in traveling the length of the spring is \( h \). Therefore, the time needed for a disturbance to travel this distance varies as \( h \). If the end effects are neglected, then the time required for the spring to traverse one step varies as its linear dimensions. If all the dimensions are doubled, the time will also be doubled, and vice versa.

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*Anybody can win unless there happens to be a second entry.*—GEORGE ADE.